Universität
Münster

Chromatic redshift for rigid categories

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MM
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Cluster of Excellence

Chromatic homotopy theory

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Remark

We now know (Burklund–Hahn–Levy–Schlank) that for all $n \geqslant 2$, $\mathrm{Sp}_{T(n)} \neq \mathrm{Sp}_{K(n)}$. For the purposes of this talk, this distinction is irrelevant.

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For commutative ring spectra, qualitative form of redshift:

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Theorems (Yuan, Burklund–Schlank–Yuan)

If R is a commutative ring spectrum of height $\geqslant 0$ and $L_{T(n)}R \neq 0$ then $L_{T(n+1)}K(R) \neq 0$.

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Together : K-theory raises the height of commutative ring spectra by exactly 1.

These results settle the “redshift *philosophy*” for commutative ring spectra, but leave many doors open:

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Arbitrary symmetric monoidal categories are too general.

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Much closer to commutative ring spectra: *rigid* stably symmetric monoidal categories (every object has a dual).

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Common slogan:

“Whatever works for commutative ring spectra works for rigid categories”

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Common slogan:

"Whatever works for commutative ring spectra works for rigid categories"

Call them "rigid categories" for this talk.

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Let C be a rigid category. There is a commutative ring map

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Noshift Theorem

Let C be a $T(n)$-local rigid category. There exists a fully faithful symmetric monoidal embedding $C \to D$ such that D is $T(n)$-local rigid, is countably generated as a C-module and

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Take away

There is no *categorical* phenomenon underlying the lower bound for redshift. The proof of Y+BSY *had* to be ring-theoretic.

Main Theorem

Let C be a rigid category. There exists a fully faithful symmetric monoidal embedding $\mathrm{C} \to \mathrm{D}$ such that D is rigid and

$$\mathrm{Dim} : \mathrm{K}(\mathrm{D}) \xrightarrow{\simeq} \mathrm{End}(\mathbf{1})^{BS^1}$$

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Corollary

Let R be an ordinary commutative ring. There exists an idempotent complete rigid stable category D with $K_0(D) \cong R$. It can be chosen to be R-linear.

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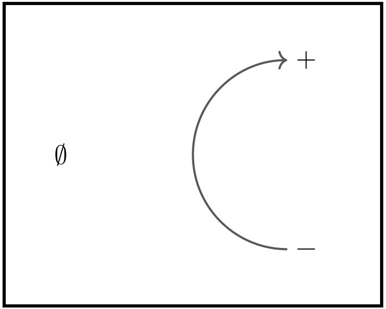
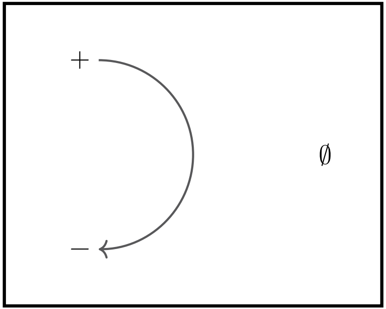
is, in this case, the circle S^1 viewed as an oriented cobordism from the empty manifold to itself



co-evaluation on $X = +$



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Proof ingredient 1

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Theorem (Barkan–Steinebrunner, R.)

Let $\mathcal{C}$ be a rigid category (+...) and $\mathcal{M}$ a $\mathcal{C}$-module. The free rigid commutative $\mathcal{C}$-algebra on $\mathcal{M}$, $\mathrm{Free}^{\mathrm{rig}}_{\mathcal{C}}(\mathcal{M})$ has endomorphism ring given by the free commutative algebra in $\mathrm{Ind}(\mathcal{C})$ on

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"Labelled" cobordisms

Relation with Dimensions

In the above context, the following square commutes :

$$\begin{array}{ccc} K(\mathcal{M})[BS^1] & \longrightarrow & K(\mathrm{Free}^{\mathrm{rig}}_{\mathcal{C}}(\mathcal{M}))[BS^1] \\ {\scriptstyle \mathrm{tr}}\downarrow & & \downarrow{\scriptstyle \widetilde{\mathrm{Dim}}} \\ \mathrm{HH}(\mathcal{M}/\mathcal{C})_{hS^1} & \xrightarrow[\text{Thm.}]{} & \mathrm{End}(\mathbb{1}_{\mathrm{Free}^{\mathrm{rig}}_{\mathcal{C}}(\mathcal{M})}) \end{array}$$

Proof ingredient 2

The previous result is how we control categorical objects (such as $\mathcal{M}$) in terms of K-theoretic information (related to $\mathrm{HH}(\mathcal{M}/\mathcal{C})$ and the Dimension morphism).

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Let $\mathcal{C}$ be a rigid category.
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Both K-theory descriptions are actually "motivic" and also hold for $\mathrm{HH}(-/\mathcal{C})$.

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(Really uses RSW and BS/R)

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For $n = 0$, both propositions are relatively easy exercises using only (a precise version of) BS/R.

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Conclude with a small object argument.

Thank you for your attention !